S.-T. Yau College Student Mathematics Contests 2022

Algebra and Number Theory

(Overall individual round)

Problem 1. Prove that $\mathbb{C}[x,y]/(x^2+y^2-1)$ is a unique factorization domain (UFD), but $\mathbb{R}[x,y]/(x^2+y^2-1)$ is not a UFD.

Problem 2. Consider the equation $f(x) = x^3 - x - 1 \in \mathbb{Z}[x]$. Let $p \neq 23$ be a prime number.

(a) Prove that $f(x) = 0 \pmod p$ has exactly one root in $\mathbb{F}_p$ if and only if $\left(\frac{p}{23}\right) = -1$.

(b) Let $K/\mathbb{Q}$ be the Galois closure corresponding to f. For p as in (a), determine the values of e, f, g in the prime decomposition of $p\mathcal{O}_K$.

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